MST121 HB AB

The Open University

USING MATHEMATICS

Handbook A and B

HANDBOOK
A and B

Copyright © 2003 The Open University
3.1

SUP 69428 6

Contents

The Greek alphabet

A	α	alpha
B	β	beta
Γ	γ	gamma
Δ	δ	delta
E	ε	epsilon
Z	ζ	zeta
H	η	eta
Θ	θ	theta
I	ι	iota
K	κ	kappa
Λ	λ	lamda
M	μ	mu
N	ν	nu
Ξ	ξ	xi
O	o	omicron
Π	π	pi
P	ρ	rho
Σ	σ	sigma
T	τ	tau
Y	υ	upsilon
Φ	ϕ	phi
X	χ	chi
Ψ	ψ	psi
Ω	ω	omega

SI units

The International System of units (SI units) is an internationally agreed set of units and symbols for measuring physical quantities.

Some of these are base units, such as

metre	symbol	m	(measurement of length),
second	symbol	s	(measurement of time),
kilogram	symbol	kg	(measurement of mass).

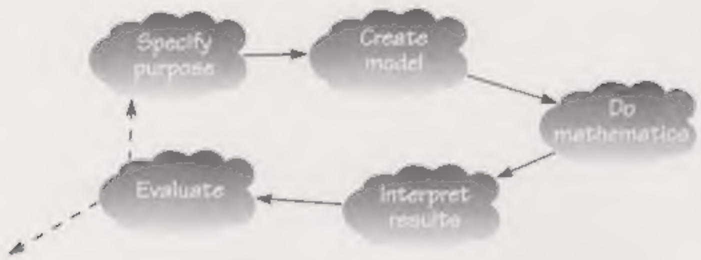
Prefixes may be added to units. Commonly used prefixes are

c	centi	or	10^{-2}	(e.g. centimetre, cm),
m	milli	or	10^{-3}	(e.g. millisecond, ms),
μ	micro	or	10^{-6}	(e.g. microsecond, μs),
k	kilo	or	10^{3}	(e.g. kilogram, kg),
M	mega	or	10^{6}	(e.g. megagram, Mg).

There are also derived units, which are used for quantities whose measurement combines base units in some way. Some of these are

area	m^2	(metres squared or square metres),
volume	m^3	(metres cubed or cubic metres),
velocity	$m\,s^{-1}$	(metres per second),
acceleration	$m\,s^{-2}$	(metres per second per second).

Mathematical modelling



The modelling process

1. Specify the purpose:
   define the problem;
   decide which aspects of the problem to investigate;
   collect relevant data.

2. Create the model:
   choose variables;
   state assumptions;
   formulate mathematical relationships.

3. Do the mathematics:
   solve equations;
   draw graphs;
   derive results.

4. Interpret the results:
   describe the mathematical solution in words;
   decide what results to compare with reality.

5. Evaluate the outcomes:
   test the outcomes of the model with reality;
   criticise the model.

Notation

Some of the notation used in the course is listed below. The right-hand column gives the chapter and page of MST121 where the symbol is first used.

$\mathbb{N}$	the set of natural numbers: $1, 2, 3, \ldots$	**A0 22**
$\mathbb{Z}$	the set of integers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$	**A0 22**
$\mathbb{Q}$	the set of rational numbers, that is, numbers of the form p/q where p and q are integers, $q \neq 0$	**A0 22**
$\mathbb{R}$	the set of real numbers	**A0 22**
$>$	greater than	**A0 24**
$<$	less than	**A0 24**
$\geq$	greater than or equal to	**A0 24**
$\leq$	less than or equal to	**A0 24**
a^n	a to the power n	**A0 25**
$\sqrt{a}$	the non-negative square root of a, where $a \geq 0$	**A0 25**
$\sqrt[n]{a}$	the nth root of a, where $a \geq 0$	**A0 25**
a^{-n}	$1/a^n$, where $a \neq 0$	**A0 25**
$a^{m/n}$	$\sqrt[n]{a^m}$ (or $(\sqrt[n]{a})^m$), where $n > 0$ and $a \geq 0$	**A3 27**
a_n	the term of a sequence with subscript n, or the sequence whose general term is a_n	**A1 7**
∞	infinity	**A1 33**
$a_n \to l$ as $n \to \infty$	used to describe the long-term behaviour of sequences; the arrow is read as 'tends to'	**A1 33**
(x, y)	the Cartesian coordinates of a point, where x is the displacement along the horizontal axis and y is the displacement along the vertical axis	**A2 6**
$\simeq$	approximately equal to	**A2 18**
$\sin^2\theta$	$(\sin(\theta))^2$; an equivalent notation is used for other powers and other trigonometric functions	**A2 34**
f	a function	**A3 6**
$f(x) = \ldots$	specifies (in terms of x) the rule of the function f	**A3 6**
$\in$	belongs to (in)	**A3 7**
$[a, b]$	the interval consisting of the set of all numbers between a and b, including a and b themselves	**A3 7**
(a, b)	the interval consisting of the set of all numbers between a and b, but not including a and b themselves	**A3 7**
$[a, b)$	the interval where the endpoint a is included but b is not; similarly $(a, b]$ includes b but not a	**A3 8**
(a, ∞)	the interval consisting of all numbers greater than a	**A3 8**
$\longmapsto$	'maps to' for variables; used to specify the rule of a function, for example, $x \longmapsto e^x$	**A3 9**
$\longrightarrow$	'maps to' for sets; used in functions to show the domain mapping to the codomain, for example, $[0, 4] \longrightarrow [0, 16]$	**A3 9**
$\lvert x \rvert$	the modulus (magnitude or absolute value) of the real number x	**A3 13**

e	the base for the natural logarithm function and the exponential function; $e = 2.718\,281\ldots$	**A3 33**
exp	the exponential function	**A3 34**
f^{-1}	the inverse function of the one-one function f	**A3 37**
$\arccos x$	the angle in the interval $[0, \pi]$ whose cosine is x	**A3 41**
$\arcsin x$	the angle in the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ whose sine is x	**A3 40**
$\arctan x$	the angle in the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ whose tangent is x	**A3 41**
$\log_a$	the logarithm function to the base a	**A3 42**
ln	the natural logarithm function, that is, $\log_e$ where $e = 2.718\,281\ldots$	**A3 44**
$\sum_{i=1}^{n} a_i$	the sum $a_1 + a_2 + \cdots + a_n$	**B1 11**
$\lim_{n\to\infty} a_n$	the limit of the convergent sequence a_n	**B1 42**
$\sum_{i=1}^{\infty} a_i$	the infinite sum $a_1 + a_2 + \cdots$	**B1 48**
$\mathbf{A}$	a matrix	**B2 19**
$\mathbf{AB}$	the product of the matrices $\mathbf{A}$ and $\mathbf{B}$	**B2 19**
$\mathbf{A}^n$	the nth power of the square matrix $\mathbf{A}$	**B2 21**
$\mathbf{A} + \mathbf{B}$	the sum of the matrices $\mathbf{A}$ and $\mathbf{B}$	**B2 22**
$k\mathbf{A}$	the scalar multiple of the matrix $\mathbf{A}$ by the real number k	**B2 23**
$\mathbf{A} - \mathbf{B}$	the difference of the matrices $\mathbf{A}$ and $\mathbf{B}$	**B2 23**
a_{ij}	the element in the ith row and jth column of the matrix $\mathbf{A}$	**B2 24**
$\mathbf{v}$	a vector	**B2 26**
v_i	the ith component of the vector $\mathbf{v}$	**B2 26**
$\mathbf{I}$	the identity matrix	**B2 35**
$\mathbf{A}^{-1}$	the inverse of the invertible matrix $\mathbf{A}$	**B2 36**
$\det \mathbf{A}$	the determinant of the square matrix $\mathbf{A}$	**B2 38**
$\mathbf{Ax} = \mathbf{b}$	the matrix form of a pair of simultaneous linear equations	**B2 41**
$\mathbf{0}$	the zero vector	**B3 8**
$\mathbf{i}$	the Cartesian unit vector $\begin{pmatrix}1\\0\end{pmatrix}$	**B3 8**
$\mathbf{j}$	the Cartesian unit vector $\begin{pmatrix}0\\1\end{pmatrix}$	**B3 8**
$\lvert\mathbf{a}\rvert$	the magnitude of the vector $\mathbf{a}$	**B3 10**
$\overrightarrow{PQ}$	the displacement vector from P to Q	**B3 12**
$\overrightarrow{OQ}$	the position vector of Q	**B3 12**
$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$	the component form of the vector $\mathbf{a}$	**B3 16**
g	the magnitude of the acceleration due to gravity	**B3 40**
$\mathbf{W}$	weight (a vector)	**B3 40**
$\mathbf{T}$	tension (a vector)	**B3 41**
$\mathbf{N}$	normal reaction (a vector)	**B3 41**

Glossary

Below is a glossary of terms used in MST121. First the definition of the term is given, then the page in this handbook where more detail can be found, and finally the chapter and page of MST121 where the term is first used.

absolute value (of a real number) See *modulus (of a real number)*.

acceleration The rate of change of velocity. **B3 40**

acceleration due to gravity The magnitude, g, of the acceleration with which an object falls. On Earth, $g = 9.8\ \mathrm{m\ s^{-2}}$. **B3 40**

arccosine The inverse function of the cosine function with domain restricted to $[0, \pi]$. 24 **A3 41**

arcsine The inverse function of the sine function with domain restricted to $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. 24 **A3 40**

arctangent The inverse function of the tangent function with domain restricted to $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. 24 **A3 41**

arithmetic mean The arithmetic mean (or average) of a set of n numbers is the sum of the numbers divided by n. **A1 14**

arithmetic sequence A sequence in which each term (apart from the first) is obtained by adding a fixed number to the previous term. 21 **A1 14**

asymptote A line which a curve approaches (arbitrarily closely) far from the origin. **A3 11**

bearing A direction given as either North or South followed by an angle (up to 90°) towards the East or West. **B3 24**

carrying capacity See *equilibrium population level*.

Cartesian coordinate system Cartesian coordinates (x, y) specify the position of a point in a plane relative to two perpendicular axes, the x-axis (horizontal) and the y-axis (vertical). **A2 6**

Cartesian unit vectors The vectors $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. 30 **B3 16**

chaotic sequence A sequence displaying apparently unstructured behaviour. 26 **B1 40**

circle The set of points in a plane that are a fixed distance from a specified point in the plane. 22 **A2 20**

closed form A formula that defines a sequence a_n in terms of the subscript n. 21 **A1 8**

codomain A set containing all the outputs of a function. See also *function*. **A3 9**

coefficient (of a term) The factor by which the term is multiplied in a particular product. **A0 30**

coefficient matrix The square matrix used when a pair of simultaneous linear equations are written in matrix form. 29 **B2 40**

column (of a matrix) See *matrix*.

common difference The difference between any two successive terms in an arithmetic sequence. 21 **A1 14**

common logarithm The logarithm function with base 10. **A3 44**

common ratio The ratio of any two successive terms in a geometric sequence. 21 **A1 19**

completed-square form The completed-square form of $x^2 + 2px$ is $(x+p)^2 - p^2$. 22 **A2 26**

component (of a vector) See *vector*.

component form (of a vector) The description of a vector $\mathbf{a}$ in terms of the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$: $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$. 30 **B3 16**

constant (1) A significant number; for example π. **A0 27**

constant (2) A term in a mathematical expression, whose value does not change during a particular calculation. **A1 14**

constant sequence A sequence in which each term has the same value. **A1 16**

continuous function Informally, a function is said to be continuous if its graph can be drawn without removing the pen from the paper. **A3 49**

continuous model A model (or representation) in which the associated quantity or quantities can vary throughout some interval of the real line. **A3 49**

continuous variable A variable that can take any value in an interval of the real line. **A3 49**

convergent sequence A sequence that settles down in the long term to values that are effectively constant. Such a sequence is said to converge. See *limit (of a sequence)*. 27 **B1 42**

cosecant (of an angle θ) The cosecant of θ is $1/\sin\theta$, provided that $\sin\theta \neq 0$. **A2 36**

cosine (of an angle θ) The first coordinate of the point P on the circumference of the unit circle, centre O, where the angle between the positive x-axis and the line segment OP is θ. By convention, angles are measured positively in an anticlockwise direction from the positive x-axis. 22 **A2 33**

Cosine Rule A rule that relates three sides and one angle of a triangle. 31 **B3 34**

cotangent (of an angle θ) The cotangent of θ is $1/\tan\theta$, provided that $\tan\theta \neq 0$. **A2 36**

cubic expression An expression of the form $ax^3 + bx^2 + cx + d$, where $a \neq 0$. **A3 47**

cycling (of a sequence) The behaviour of a sequence that takes a number of different but repeating values; for example, in a 2-cycle the sequence settles to a pattern of alternating between two values. 26 **B1 40**

decreasing function A real function f with the property that for all x_1, x_2 in the domain of f, if $x_1 < x_2$, then $f(x_1) > f(x_2)$. 24 **A3 36**

degree of a polynomial For a polynomial in x, the highest power of x with a non-zero coefficient. **A3 47**

dependent variable A variable whose value depends on the value of another variable (or variables). A0 27

determinant (of a matrix) The determinant of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$. 29 B2 38

direction (of a vector) The angle θ, measured anticlockwise, that an arrow representing the vector makes with the positive x-direction. 30 B3 11

discrete model A model (or representation) in which the associated quantity or quantities can only take separated values. A3 49

discrete variable A variable that can only take on values in a separated set, such as the integers. A3 49

displacement (vector) The displacement vector from P to Q is represented by the arrow with its tail at P and its tip at Q. B3 12

domain (of a function) The set of allowable input values for a function. See also *function*. 23 A3 6

element (of a matrix) See *matrix*.

equal matrices Two matrices are equal if they are of the same size and all their corresponding elements agree. B2 18

Equilibrium Condition for forces The relationship between the forces acting on an object at rest. 31 B3 42

equilibrium population level The population size at which the proportionate growth rate is zero; that is, the size at which the population remains constant. It is represented by the parameter E in the logistic recurrence relation. 26 B1 32

exponential function A function with domain $\mathbb{R}$ and rule of the form $f(x) = a^x$ for some positive real number a. The number a is called the base of the exponential function. The most important exponential function is $f(x) = e^x$, where $e = 2.718\,281\ldots$. The function $f(x) = e^x$ is also written as $\exp(x)$. 25 A3 32

exponential model A model for population variation, based on the assumption of a constant proportionate growth rate, r. The model is described by the recurrence relation $P_{n+1} = (1+r)P_n$, where P_n is the population at n years after some chosen starting time. 26 B1 22

finite decimal A number whose decimal representation has a finite number of decimal places. A0 22

finite geometric series A sum of the form $a + ar + ar^2 + \cdots + ar^n$. 22 A1 28

finite sequence A sequence with a finite number of terms. A1 6

force A push or pull which, if not counteracted, causes the acceleration of an object. B3 39

force diagram A diagrammatic representation of the forces acting on an object. B3 41

function A (real) function consists of a subset of $\mathbb{R}$, called the domain, and a rule that associates with each x in the domain a unique y in $\mathbb{R}$. A function may also be referred to as a *mapping* or a *transformation*. 23 A3 6

geometric form (of a vector) The description of a vector $\mathbf{a}$ in terms of its magnitude $\lvert\mathbf{a}\rvert$ and its direction θ.	30	B3 11
geometric mean The geometric mean of a set of n positive numbers is the nth root of the product of the numbers.		A1 19
geometric sequence A sequence in which each term (apart from the first) is obtained by multiplying the previous term by a fixed number.	21	A1 19
geometric series See *finite geometric series* and *infinite geometric series*.		
gradient (of a line) See *slope (of a line)*.		
graph (of a real function f) The set of points $(x, f(x))$ in the Cartesian plane.		A3 9
hypotenuse The longest side of a right-angled triangle.		A2 21
i-component (of a vector a) The number a_1 in the component form of the vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$.	30	B3 16
identity matrix A square matrix with all the elements on its leading diagonal equal to 1 and all the other elements equal to 0.	29	B2 35
image (of x under f) The output of the function f for a given input x, that is, the value of $f(x)$.	23	A3 7
image set The complete set of output values of a function.	23	A3 12
increasing function A real function f with the property that for all x_1, x_2 in the domain of f, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.	24	A3 36
independent variable A variable that can take any value appropriate to the problem; that is, its value does not depend on the value of any other variable.		A0 27
infinite decimal A number whose decimal representation has infinitely many decimal places.		A0 22
infinite geometric series A sum of the form $a + ar + ar^2 + \cdots$.	26	B1 48
infinite sequence A sequence that has a first term but no final term.		A1 6
integers The positive and negative whole numbers, together with zero: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.		A0 22
intercept A value of x or y where a line (or curve) meets the x-axis or y-axis, respectively. The x-intercept is the value of x where it meets the x-axis. The y-intercept is the value of y where it meets the y-axis.	24	A2 11
interval An unbroken subset of the real line.		A3 7
inverse function The inverse function f^{-1} of a one-one function f reverses the effect of f; that is, if $f(x) = y$, then $f^{-1}(y) = x$. The domain of f^{-1} is the image set of f.	24	A3 37
inverse of a matrix If two square matrices $\mathbf{A}$ and $\mathbf{B}$ have the property that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix, then each of $\mathbf{A}$ and $\mathbf{B}$ is the inverse of the other.	29	B2 36
invertible matrix A square matrix that has a non-zero determinant, and therefore has an inverse.	29	B2 38

irrational number A real number that is not rational, and hence is a non-recurring decimal. A0 23

j-component (of a vector a) The number a_2 in the component form of the vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$. 30 B3 16

leading diagonal (of a matrix) The diagonal of a square matrix which starts at the top left and ends at the bottom right of the matrix. B2 36

limit (of a sequence) The value near which a convergent sequence settles in the long term. 27 B1 42

linear expression A sum consisting of first powers of the variables and a constant. A0 34

linear function A real function with rule of the form $f(x) = mx + c$, for some constants m and c. A3 9

linear recurrence sequence A recurrence sequence with recurrence relation of the form $x_{n+1} = rx_n + d$, where r and d are constants. 21 A1 24

locus A curve defined by a particular property. A2 7

logarithm to the base a The inverse function of the exponential function $f(x) = a^x$, where $a > 0, a \neq 1$. The logarithm function is written $\log_a$ and has domain $(0, \infty)$. 25 A3 42

logistic model A model for population variation, based on the assumption of a proportionate growth rate of the form $R(P) = r(1 - P/E)$, where r and E are positive parameters. 26 B1 29

logistic recurrence relation A recurrence relation of the form $P_{n+1} - P_n = rP_n(1 - P_n/E)$, where r and E are positive parameters. 26 B1 29

long-term behaviour of a sequence The way in which the sequence develops as more and more terms are considered. 21 A1 32

magnitude (of a real number) See *modulus (of a real number)*.

magnitude (of a vector a) The length of an arrow representing the vector; it is written as $|\mathbf{a}|$. 30 B3 10

main diagonal (of a matrix) See *leading diagonal (of a matrix)*.

many-one function A function that is not one-one. A3 36

mapping See *function*.

mass A measure of the amount of matter that an object contains. B3 40

mathematical model A collection of formulas that attempts to quantify how some aspect of the real world behaves. 6 A1 36

$m \times n$ **matrix** A matrix that has m rows and n columns. 28 B2 18

matrix A rectangular array of numbers. Each number in a matrix is called an element of the matrix. A row of the matrix is a horizontal line of numbers in the array, and a column of the matrix is a vertical line of numbers in the array. 28 B2 11

matrix addition Two matrices of the same size are added by adding their corresponding elements. 28 B2 21

matrix multiplication Two matrices **A** and **B** can be multiplied only if the number of columns in **A** is equal to the number of rows in **B**. The corresponding product matrix **AB** is a matrix of elements formed by combining, in turn, each row of **A** with each column of **B**. Combining row i of **A** with column j of **B** gives the ijth element of **AB**. 28 **B2 16**

matrix subtraction Two matrices of the same size are subtracted by subtracting their corresponding elements. **B2 23**

matrix–vector multiplication The process of multiplying a vector by a matrix. See *matrix multiplication*. 28 **B2 13**

mean An average of a finite set of numbers. See *arithmetic mean*, *geometric mean*.

modelling cycle The process of choosing a (mathematical) model, trying it out, evaluating it, and possibly changing it. 6 **A1 37**

modulus (of a real number x) The magnitude of x, regardless of its sign; it is written as $|x|$. 23 **A3 13**

natural logarithm The logarithm function with base e, where $e = 2.718\,281\ldots$; it is often written as ln. 25 **A3 45**

natural numbers The positive integers: $1, 2, 3, \ldots$. **A0 22**

network diagram A mathematical representation of a physical network. Each point at which a network branches is called a node, and two nodes of a network may be connected by a pipe. 27 **B2 6**

newton The SI unit of force. **B3 40**

node (of a network) See *network diagram*.

non-invertible matrix A square matrix that has zero determinant, and therefore has no inverse. 29 **B2 38**

normal reaction The force acting on an object due to contact with a surface, which is directed at right angles to that surface. **B3 41**

nth root of a The non-negative number which, when raised to the power n, gives the answer a. 18 **A0 25**

one-one function A function f with the property that for all x_1, x_2 in the domain of f, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. 24 **A3 36**

parabola The graph of a quadratic function. **A3 19**

Parallelogram Rule A rule for the addition of two vectors that are in geometric form. 30 **B3 14**

parameter (1) A variable used when defining a family of mathematical objects, such as a recurrence system. 21 **A1 14**

parameter (2) A variable, often t, used when defining a curve in terms of the motion of a point along the curve. 23 **A2 40**

parametrisation (of a curve) The process of describing the coordinates of points on the curve in terms of a parameter. 23 **A2 40**

particle A material object whose size and internal structure may be neglected (for modelling purposes). **B3 40**

periodic function A function f whose graph is unchanged by a horizontal translation through the period p; that is, $f(x+p) = f(x)$ for all x in the domain. A3 28

perpendicular bisector (of a line segment AB) The line that cuts AB halfway along its length and is at right angles to AB. A2 24

pipe (of a network) See *network diagram*.

polar coordinates (of a point A) The numbers r and θ for the point A with Cartesian coordinates $(r\cos\theta, r\sin\theta)$. B3 19

polynomial A polynomial (of degree n) is an expression of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$. A3 47

polynomial function A function whose rule is a polynomial; for example, a quadratic function. A3 47

position vector A displacement vector whose arrow has its tail at the origin. B3 12

power (of a square matrix) The nth power of a square matrix $\mathbf{A}$, written $\mathbf{A}^n$, is obtained by repeated matrix multiplication. 28 B2 21

product matrix The result of multiplying two matrices together. B2 16

proportionate growth rate The proportionate birth rate minus the proportionate death rate for a population. B1 22

quadratic expression An expression in the form $ax^2 + bx + c$, where $a \neq 0$. A0 36

radius (of a circle) The distance from the centre of a circle to any point on the circumference of the circle. A2 20

rational number A real number that can be represented as a fraction, and hence as a recurring decimal. A0 22

real function A function for which both the inputs and the outputs are real numbers. A3 9

real line A number line that includes all real numbers. A0 23

real number A number that can be represented as a decimal. A0 22

reciprocal function The function whose rule is $f(x) = 1/x$. A3 10

recurrence relation A formula that defines each term of a sequence by referring to a previous term or terms of the sequence. 21 A1 11

recurrence system The specification of a sequence by an initial term or terms, a recurrence relation and a subscript range. 21 A1 11

resultant The sum of two vectors. B3 13

rise from A to B The rise from a point $A(x_1, y_1)$ to a point $B(x_2, y_2)$ is $y_2 - y_1$. A2 9

row (of a matrix) See *matrix*.

rule (of a function) The process for converting each input value in the domain of the function into a unique output value. See *function*. 23 A3 6

run from A to B The run from a point $A(x_1, y_1)$ to a point $B(x_2, y_2)$ is $x_2 - x_1$. A2 9

scalar A real number. B2 23

scalar multiplication (of a matrix) The operation of multiplying each element of a matrix $\mathbf{A}$ by a real number k. The resulting matrix $k\mathbf{A}$ is called the scalar multiple of $\mathbf{A}$ by the real number k. 28 B2 23

scaling (of a graph) Squashing or stretching the graph in the x-direction or y-direction. 24 A3 21

secant (of an angle θ) The secant of θ is $1/\cos\theta$, provided that $\cos\theta \neq 0$. A2 36

sequence An ordered list (finite or infinite) of numbers, called the terms of the sequence. 21 A1 6

series A sum of consecutive terms of a sequence. 26 B1 11

sigma notation A concise way of expressing finite and infinite series. 26 B1 11

sine (of an angle θ) The second coordinate of the point P on the circumference of the unit circle, centre O, where the angle between the positive x-axis and the line segment OP is θ. By convention, angles are measured positively in an anticlockwise direction from the positive x-axis. 22 A2 33

Sine Rule A rule that relates pairs of sides and the corresponding opposite angles of a triangle. 31 B3 31

size (of a matrix) An $m \times n$ matrix has size $m \times n$. 28 B2 18

slope (of a line) If A and B are two points on a line, then the slope of the line is (rise from A to B) $\div$ (run from A to B). The slope is also called the gradient. 22 A2 9

solving a triangle The process of determining all the angles and side lengths of a triangle. A2 37

speed A (scalar) measure of how fast an object is moving, irrespective of its direction of motion. The speed is the magnitude of the velocity vector. B3 26

square matrix A matrix with the same number of rows and columns. B2 21

subscript In the notation a_n, n is called the subscript of a. A1 7

subtended angle The angle at a point, when the angle lies between two lines drawn from the point to the endpoints of a line segment or an arc of a circle. A2 21

tangent (of an angle θ) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, where $\theta \neq \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \ldots$. 22 A2 36

tangent (to a circle) A line that intersects the circle in precisely one point. A2 29

tends to l The terms of a sequence become arbitrarily close to l. A1 33

tends to infinity (or minus infinity) The terms of a sequence become arbitrarily large and positive (or arbitrarily large and negative). A1 33

tension The force provided by a taut string (rope, etc.). B3 41

term (of a sequence) An item of a sequence. 21 A1 6

transformation See *function.*		
translation (of a graph) Shifting the graph horizontally or vertically.	24	**A3 20**
triangle of forces A triangle of arrows which is a geometric representation of the Equilibrium Condition for three forces.		**B3 44**
Triangle Rule A rule for the addition of two vectors that are in geometric form.	30	**B3 13**
unbounded sequence A sequence that has terms of arbitrarily large value, either positive or negative.	21	**A1 34**
unit circle The circle with radius 1 and centre at the origin.		**A2 33**
variable A symbol used to represent a quantity that can vary.		**A0 27**
vector A matrix consisting of a single column. Each number appearing in the column is called a component of the vector.	28	**B2 10**
velocity (vector) The rate of change of position of an object. Velocity is a measure of how fast the object is moving and its direction of motion.		**B3 26**
weight The force on an object due to gravity.		**B3 40**
zero vector The vector in which every component is 0, denoted by **0**.		**B3 8**

Background material for MST121

Rounding numbers

To **round** to a given number of **decimal places**, look at the digit one place to the right of the number of places specified. If this digit is 5 or more, then round up; if it is less than 5, then round down.

To **round** to a given number of **significant figures**, start counting significant figures from the first non-zero digit on the left, and follow the rules for rounding.

Scientific notation

In scientific notation, positive numbers are expressed in the form $a \times 10^n$, where a is between 1 and 10, and n is an integer.

Rules for powers

$a^p \times a^q = a^{p+q}$ $\quad a^p \div a^q = a^{p-q}$ $\quad (a^p)^q = a^{pq}$ $\quad a^p b^p = (ab)^p$

$a^{-n} = \dfrac{1}{a^n}$ $\quad a^0 = 1$ $\quad a^{1/n} = \sqrt[n]{a}$ $\quad a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Calculating means

To calculate the **mean** of a batch of data, add together the values (x) in the batch to give $\sum x$, and divide by n, the number of values in the batch.

Algebra

Difference of two squares: $a^2 - b^2 = (a-b)(a+b)$.

Squaring a bracket: $(a+b)^2 = a^2 + 2ab + b^2$.

The **solutions of the quadratic equation** $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To solve two simultaneous linear equations by **substitution**.

1. Rearrange one of the equations so that one unknown is equal to an expression involving the other unknown.
2. Substitute this expression in the other equation.
3. Solve the resulting linear equation for the other unknown.
4. Substitute this solution into either of the original equations to find the remaining unknown.

To solve two simultaneous linear equations by **elimination**.

1. Multiply the two equations by numbers chosen so that one of the unknowns has the same coefficient, possibly with the opposite sign, in both equations.
2. Subtract or add the new equations to eliminate that unknown.
3. Solve the resulting linear equation for the other unknown.
4. Substitute this solution into either of the original equations to find the remaining unknown.

Equivalent rearrangements of inequalities

If the sides of an inequality are interchanged, then the direction of the inequality sign is reversed: $>$ becomes $<$, $\geq$ becomes $\leq$, and vice versa.

The same number can be added to (or subtracted from) both sides of an inequality: if $a < b$, then $a + c < b + c$.

Both sides of an inequality can be multiplied (or divided) by the same positive number: if $a < b$ and $c > 0$, then $ac < bc$.

If both sides of an inequality are multiplied (or divided) by the same negative number, then the direction of the inequality sign changes: if $a < b$ and $c < 0$, then $ac > bc$.

Angle measurement

The angle subtended at the centre of a circle by an arc equal in length to the radius of the circle is defined to be one **radian**. Thus 2π radians = 360°, and the rules for converting between degrees and radians are

$$x \text{ radians} = x \times \frac{180}{\pi} \text{ degrees}, \quad y \text{ degrees} = y \times \frac{\pi}{180} \text{ radians}.$$

Polygons

A plane figure which is a closed shape whose sides are straight lines is called a **polygon**. A point where two sides meet is called a **vertex**. A polygon with n sides (and hence n vertices) is referred to as an **n-gon**.

The angle sum of an n-gon is $(n-2)180°$, that is, $(n-2)\pi$ radians.

An n-gon is said to be **regular** if all its sides are equal and all its angles are equal.

Triangles

A **triangle** is a polygon with three sides. Its angle sum is 180°, that is, π radians. If all three sides are of equal length, then it is an **equilateral triangle** and all three angles are 60°. If two sides are of equal length, then it is an **isosceles triangle** and the two angles opposite the equal length sides are equal.

The **area of a triangle** is

- $\frac{1}{2}ah$, where a is the length of the base and h is the height;
- $\frac{1}{2}ab\sin\theta$, where a and b are two side lengths, and θ is the angle between the sides.

Right-angled triangles

Pythagoras' Theorem: For a triangle ABC (with side lengths a, b and c opposite A, B and C, respectively), where the angle at C is a right angle, $c^2 = a^2 + b^2$.

The side opposite the right angle is known as the **hypotenuse**.

For this triangle, the trigonometric ratios are

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}, \quad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}, \quad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b},$$

$$\operatorname{cosec} A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{a}, \quad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{b}, \quad \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a}.$$

It follows from Pythagoras' Theorem that $\sin^2 A + \cos^2 A = 1$.

Useful trigonometic ratios

Angle θ in radians	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$
Angle θ in degrees	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	Undefined

Sine positive	All positive
Tangent positive	Cosine positive

Quadrilaterals

A **quadrilateral** is a polygon with four sides. Its angle sum is 360°, that is, 2π radians.

- A quadrilateral in which opposite sides are equal is a **parallelogram**.
- A quadrilateral in which all sides are equal is a **rhombus**.
- A quadrilateral in which all angles are equal (to 90°) is a **rectangle**.
- A quadrilateral in which all sides and all angles are equal is a **square**.

In a parallelogram, opposite angles are equal and the two diagonals bisect each other. In a rhombus, the diagonals bisect each other at an angle of 90°.

The area of a rectangle is $A = lb$, where l is the length and b is the breadth.

Circles

A circle of radius r has

- circumference $C = 2\pi r = \pi d$, where d is the diameter;
- area $A = \pi r^2$.

Congruence

Two figures are **congruent** if they have the same shape and the same size.

Two n-gons are congruent if all corresponding sides and angles are equal.

Similarity

Two figures are **similar** if they have the same shape; their sizes need not be the same.

Two n-gons are similar if each angle in one n-gon is equal to the corresponding angle in the other. In this case, the length of each side in one n-gon is the same multiple of the corresponding length in the other.

Prisms

A **prism** is a solid with constant cross-section. A **cylinder** is a prism with circular cross-section.

The surface area of a prism is the sum of the areas of its faces. In particular, the surface area of a cylinder is $A = 2\pi r^2 + 2\pi rh$, where r is the radius of the circular cross-section and h is the length.

The volume of a prism is the area of its cross-section multiplied by its length. In particular, the volume of a cylinder is $V = \pi r^2 h$.

Definitions and results in MST121

The following definitions and results have been collected from the chapters of MST121. They are listed in chapter order. If you cannot find the information you want here, then try the alphabetical listing in the Glossary.

MST121 Chapter A1 Sequences

Types of sequences

Convention: The first term of a sequence has subscript 1, unless otherwise indicated.

An **arithmetic sequence** with first term a and common difference d can be specified by either of the following recurrence systems:

◇ $$x_1 = a, \quad x_{n+1} = x_n + d \quad (n = 1, 2, 3, \ldots),$$

with closed form $x_n = a + (n-1)d$ $(n = 1, 2, 3, \ldots)$;

◇ $$x_0 = a, \quad x_{n+1} = x_n + d \quad (n = 0, 1, 2, \ldots),$$

with closed form $x_n = a + nd$ $(n = 0, 1, 2, \ldots)$.

A **geometric sequence** with first term a and common ratio r can be specified by either of the following recurrence systems:

◇ $$x_1 = a, \quad x_{n+1} = rx_n \quad (n = 1, 2, 3, \ldots$$

with closed form $x_n = ar^{n-1}$ $(n = 1, 2, 3,$

◇ $$x_0 = a, \quad x_{n+1} = rx_n \quad (n = 0, 1, 2, \ldots),$$

with closed form $x_n = ar^n$ $(n = 0, 1, 2, \ldots)$.

A **linear recurrence sequence** with parameters a, r and d can be specified by either of the following recurrence systems:

◇ $$x_1 = a, \quad x_{n+1} = rx_n + d \quad (n = 1, 2, 3, \quad),$$

with closed form (when $r \neq 1$)

$$x_n = \left(a + \frac{d}{r-1}\right) r^{n-1} - \frac{d}{r-1} \quad (n = 1, 2, 3, \ldots);$$

◇ $$x_0 = a, \quad x_{n+1} = rx_n + d \quad (n = 0, 1, 2, \ldots),$$

with closed form (when $r \neq 1$)

$$x_n = \left(a + \frac{d}{r-1}\right) r^n - \frac{d}{r-1} \quad (n = 0, 1, 2, \ldots).$$

Long-term behaviour of sequences

Range of r	Behaviour of r^n
$r > 1$	$r^n \to \infty$ as $n \to \infty$
$r = 1$	Remains constant: $1, 1, 1, \ldots$
$0 < r < 1$	$r^n \to 0$ as $n \to \infty$
$r = 0$	Remains constant: $0, 0, 0, \ldots$
$-1 < r < 0$	$r^n \to 0$ as $n \to \infty$, alternates in sign
$r = -1$	Alternates between -1 and $+1$
$r < -1$	r^n is unbounded, alternates in sign

Sum of a finite geometric series

The sum of a finite geometric series with first term a and common ratio r ($r \neq 1$) is

$$a + ar + ar^2 + \cdots + ar^n = a\left(\frac{1 - r^{n+1}}{1 - r}\right).$$

MST121 Chapter A2 Lines and circles

Lines

Type	Slope	Equation
Parallel to x-axis	$m = 0$	$y = c$, where c is a constant
Parallel to y-axis	Infinite	$x = d$, where d is a constant
Not parallel to y-axis	$m = (y_2 - y_1)/(x_2 - x_1)$, where (x_1, y_1) and (x_2, y_2) are points on the line	$y - y_1 = m(x - x_1)$ or $y = mx + c$, where c is the y-intercept

If two lines are **parallel**, then they have equal slopes. If two lines are **perpendicular**, then either the product of their slopes is -1 or one has slope 0 and the other has infinite slope.

The **distance** between two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The **midpoint** of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Circles

Geometrically, a **circle** is the set of points that are at a fixed distance (the radius) from a specified point (the centre). Algebraically, a circle with centre (a, b) and radius r has the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

To find the equation of a circle, given three points A, B and C on the circle, find the perpendicular bisectors of the line segments AB and BC. The centre of the circle is the intersection point of the two perpendicular bisectors. The radius of the circle is the distance from the centre to any of the points A, B or C.

To **complete the square** of $x^2 + 2px$, use

$$x^2 + 2px = x^2 + 2px + p^2 - p^2 = (x + p)^2 - p^2.$$

Trigonometry

Let $P(x, y)$ be a point on the unit circle (with centre O) such that the angle from the positive x-axis to OP is θ (measured anticlockwise if θ is positive, clockwise if θ is negative). Then

$$\cos\theta = x, \qquad \sin\theta = y \quad \text{and} \quad \tan\theta = \frac{\sin\theta}{\cos\theta} \quad (\text{provided that } \cos\theta \neq 0).$$

Trigonometric identities

$$\cos(-\theta) = \cos\theta, \qquad \cos(\pi - \theta) = -\cos\theta, \qquad \cos(\theta + 2\pi) = \cos\theta,$$
$$\sin(-\theta) = -\sin\theta, \qquad \sin(\pi - \theta) = \sin\theta, \qquad \sin(\theta + 2\pi) = \sin\theta,$$
$$\tan(-\theta) = -\tan\theta, \qquad \tan(\pi - \theta) = -\tan\theta, \qquad \tan(\theta + \pi) = \tan\theta,$$
$$\cos(\tfrac{1}{2}\pi - \theta) = \sin\theta, \qquad \sin(\tfrac{1}{2}\pi - \theta) = \cos\theta, \qquad \cos^2\theta + \sin^2\theta = 1.$$

Parametrisation of lines and circles

- A line with slope m passing through the point (x_1, y_1) has parametric equations

$$x = t + x_1, \quad y = mt + y_1.$$

- A line passing through the two points (x_1, y_1) and (x_2, y_2) has parametric equations

$$x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1).$$

- A circle with centre at (a, b) and radius r has parametric equations

$$x = a + r\cos\theta, \quad y = b + r\sin\theta \quad (0 \le \theta \le 2\pi).$$

MST121 Chapter A3 Functions

Functions

A **(real) function** is specified by giving

- the **domain**, that is, the set of allowable input values, which are real numbers;
- the **rule** for converting each input value to a unique output value, which is also a real number.

The output of a function f for a given input x is called the **image** of x under f, and is written $f(x)$. The set of all outputs of the function f is called the **image set** of f.

Convention: When a function is specified just by a rule, it is understood that the domain of the function is the largest possible set of real numbers for which the rule is applicable.

Function notation

A standard notation used to specify a function f is

$$f(x) = x^2 + 1 \quad (0 \le x \le 6).$$

Other notations used to specify the same function f are

- $f : x \longmapsto x^2 + 1 \quad (0 \le x \le 6)$;
- $f : [0, 6] \longrightarrow \mathbb{R}$
  $x \longmapsto x^2 + 1$

Modulus of a real number

The **modulus**, or **absolute value**, of a real number x is the magnitude of x, regardless of sign, denoted by $|x|$. Thus

$$|x| = \begin{cases} x, & \text{if } x \ge 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Translating and scaling a known graph

Graph	Translation or scaling of $y = f(x)$
$y = f(x + p)$	Horizontal translation by p units to the left (right if p is negative)
$y = f(x) + q$	Vertical translation by q units upwards (downwards if q is negative)
$y = af(x)$	y-scaling with factor a
$y = f(bx)$	x-scaling with factor $1/b$

Graphing quadratic functions

To sketch the graph of a quadratic function $f(x) = ax^2 + bx + c$, first write the function in completed-square form: $f(x) = a(x + p)^2 + q$. Then start with the graph of $y = x^2$ and perform

1. a y-scaling with factor a;
2. a horizontal translation by p units to the left (right if p is negative);
3. a vertical translation by q units upwards (downwards if q is negative).

The graph is a parabola with vertex $(-p, q)$, which is the lowest point if $a > 0$ and the highest point if $a < 0$. Its axis of symmetry is $x = -p$.

The y-intercept is found by setting $x = 0$ to give $f(0) = c$. The x-intercepts (if any) are found by setting $y = 0$ and solving $ax^2 + bx + c = 0$

Inverse functions

A real function f is **one-one** if it has the following property: for all x_1, x_2 in the domain of f, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

A real function f is **increasing** if it has the following property: for all x_1, x_2 in the domain of f, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.

A real function f is **decreasing** if it has the following property: for all x_1, x_2 in the domain of f, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

If a function is increasing (or decreasing), then it is one-one.

When a function f is one-one, an **inverse function** f^{-1} can be defined which reverses the action of f.

- ◇ To obtain the rule for the function f^{-1} (in terms of x), solve the equation $y = f(x)$ to obtain x in terms of y, and then exchange the roles of x and y. The image set of f is the domain of f^{-1}, and vice versa.
- ◇ To obtain the graph of $y = f^{-1}(x)$, reflect the graph of $y = f(x)$ in the 45° line.

Inverse trigonometric functions

The function $f(x) = \sin x$ $(-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi)$ has an inverse function **arcsine** with domain $[-1, 1]$ and image set $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Thus, for $-1 \leq y \leq 1$,

$$x = \arcsin y \quad \text{means that} \quad y = \sin x \text{ and } -\tfrac{1}{2}\pi \leq x \leq \tfrac{1}{2}\pi.$$

The function $f(x) = \cos x$ $(0 \leq x \leq \pi)$ has an inverse function **arccosine** with domain $[-1, 1]$ and image set $[0, \pi]$. Thus, for $-1 \leq y \leq 1$,

$$x = \arccos y \quad \text{means that} \quad y = \cos x \text{ and } 0 \leq x \leq \pi.$$

The function $f(x) = \tan x$ $(-\frac{1}{2}\pi < x < \frac{1}{2}\pi)$ has an inverse function **arctangent** with domain $(-\infty, \infty)$ and image set $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. Thus, for $-\infty < y < \infty$,

$$x = \arctan y \quad \text{means that} \quad y = \tan x \text{ and } -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi.$$

Logarithms

An exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, has domain $\mathbb{R}$ and image set $(0, \infty)$. Its inverse function, called **logarithm** to the **base** a and denoted by $\log_a$, has domain $(0, \infty)$ and image set $\mathbb{R}$. Thus, for $y > 0$,

$$x = \log_a y \quad \text{means that} \quad y = a^x.$$

The **natural logarithm** has base $e = 2.718\,281\ldots$ and is often written as ln. The **common logarithm** has base 10 and is often written as log.

Provided that $a > 0$ and $a \neq 1$, the logarithm to the base a has the following properties:

(a) $\log_a 1 = 0$, $\log_a a = 1$;

(b) for $x > 0$ and $y > 0$,

  (i) $\log_a(xy) = \log_a x + \log_a y$,

  (ii) $\log_a(x/y) = \log_a x - \log_a y$;

(c) for $x > 0$ and p in $\mathbb{R}$, $\quad \log_a(x^p) = p \log_a x$.

To use logarithms to solve an equation of the form $a^x = k$, where $k > 0$ and $a > 0, a \neq 1$, apply the function ln to both sides of the equation, and use property (c) to obtain

$$x = \frac{\ln k}{\ln a}.$$

Formulas for sums

$$\sum_{i=1}^{n} c_i = c_1 + c_2 + \cdots + c_n$$

$$\sum_{i=0}^{n} ar^i = a + ar + ar^2 + \cdots + ar^n = a\left(\frac{1-r^{n+1}}{1-r}\right) \quad (r \neq 1)$$

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r} \quad (|r| < 1)$$

$$\sum_{i=1}^{n}(a + bx_i) = an + b\sum_{i=1}^{n} x_i \qquad \sum_{i=m}^{n}(a + bx_i) = a(n - m + 1) + b\sum_{i=m}^{n} x_i$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1)$$

Exponential model

The exponential model for population variation is based on the assumption of a constant proportionate growth rate, r. The model is described by either the recurrence relation

$$P_{n+1} = (1 + r)P_n \quad (n = 0, 1, 2, \ldots),$$

or its closed-form solution

$$P_n = (1 + r)^n P_0 \quad (n = 0, 1, 2, \ldots),$$

where P_n is the population size at n years after some chosen starting time. The proportionate growth rate r is the proportionate birth rate minus the proportionate death rate.

Logistic model

The logistic model for population variation is based on the assumption of a proportionate growth rate $R(P)$ of the form $R(P) = r(1 - P/E)$, where r and E are positive parameters. The model is described by the recurrence relation

$$P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right) \quad (n = 0, 1, 2, \ldots),$$

where P_n is the population size at n years after some chosen starting time. The positive constant r represents the proportionate growth rate of the population when the population size is small, and the positive constant E represents the equilibrium population level (the population size at which the proportionate growth rate is zero).

The long-term behaviour of sequences generated by the logistic recurrence relation (with $0 < P_0 < E(1 + 1/r)$) depends on the value of r, as shown in the table below.

Range of r	Long-term behaviour of P_n
$0 < r \leq 1$	Settles close to (converges to) E, with values always just below E
$1 < r < 2$	Settles close to E, with values alternating between just above and just below E
$2 < r \leq 2.44$	2-cycle, with one value above E and one value below E
$2.45 \leq r \leq 2.54$	4-cycle, with two values above E and two values below E
$2.6 \leq r \leq 3$	Chaotic variation between bounds (with some exceptions)

Convergence and limits

If a sequence P_n settles down in the long term to values that are effectively constant, then P_n is said to be **convergent**. The value l near which P_n settles in the long term is called the **limit** of the sequence, written

$$\lim_{n \to \infty} P_n = l.$$

If a sequence P_n, given by a recurrence relation, converges, then it converges to one of the values that generate a constant sequence from the recurrence relation. To find such values, substitute $P_n = l$ and $P_{n+1} = l$ into the recurrence relation and solve for l.

Reciprocal Rule

If the terms of a sequence b_n are of the form $1/a_n$, where terms of the sequence a_n become arbitrarily large as n increases, then the sequence b_n is convergent, and $\lim_{n \to \infty} b_n = 0$.

Constant Multiple Rule

If the terms of a sequence b_n are of the form ca_n, where the sequence a_n is convergent with limit 0, and c is a constant, then the sequence b_n is also convergent, and $\lim_{n \to \infty} b_n = 0$.

Long-term 'basic sequence' behaviour

The long-term behaviour of the sequence r^n ($n = 1, 2, 3, \ldots$) is as follows.

- If $|r| > 1$, then $|r^n| \to \infty$ as $n \to \infty$. (If $r > 1$, then $r^n \to \infty$ as $n \to \infty$; if $r < -1$, then r^n is unbounded and alternates in sign.)
- If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$.
- If $r = 1$, then $r^n = 1$. If $r = -1$, then r^n alternates between 1 and -1.

The long-term behaviour of the sequence n^p ($n = 1, 2, 3, \ldots$) is as follows.

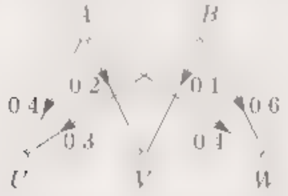
- If $p > 0$, then $n^p \to \infty$ as $n \to \infty$.
- If $p < 0$, then $n^p \to 0$ as $n \to \infty$.
- If $p = 0$, then $n^p = 1$.

MST121 Chapter B2 Modelling with matrices

Networks and matrices

A physical network can be represented by a **network diagram** and also by a matrix. The entries of the matrix indicate the proportion of the input flowing through each pipe of the network.

The network diagram and the matrix shown below represent the same physical network having two input nodes A and B, and three output nodes U, V and W.



$$\begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \\ 0.4 & 0.6 \end{pmatrix}$$

If the outputs of one network feed directly into an equal number of inputs in a second network, then the matrix representing the combined network is obtained by multiplying the matrices representing the two original networks.

Arithmetic of matrices and vectors

A **matrix** is a rectangular array of numbers. Each number in a matrix is called an **element**. A matrix with m rows and n columns is called an $m \times n$ matrix, and has **size** $m \times n$. The element in row i and column j of a matrix $\mathbf{A}$ is written as a_{ij}.

Matrix multiplication

Two matrices $\mathbf{A}$ and $\mathbf{B}$ can be multiplied only if the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$. The element in the ith row and jth column of the product matrix $\mathbf{AB}$ is obtained by adding up the products of corresponding elements of the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$.

Thus, if $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix, then $\mathbf{C} = \mathbf{AB}$ is an $m \times p$ matrix with elements given by

$$c_{ij} = \sum_{k=1} a_{ik}b_{kj} \quad (i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, p).$$

For example, if $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$, then

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{pmatrix}$$

In most cases, $\mathbf{AB} \neq \mathbf{BA}$. The **power** $\mathbf{A}^n$ of a square matrix $\mathbf{A}$ is formed by multiplying together n matrices $\mathbf{A}$; for example, $\mathbf{A}^3 = \mathbf{AAA}$.

Matrix addition

Two matrices $\mathbf{A}$ and $\mathbf{B}$ can be added only if they have the same size. If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices, then $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is also an $m \times n$ matrix with elements given by

$$c_{ij} = a_{ij} + b_{ij} \quad (i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n).$$

For example, if $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$, then

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{pmatrix}$$

Scalar multiplication

When a matrix is scalar multiplied by a real number k, each element of the matrix is multiplied by k. For example, if

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, \text{ then } k\mathbf{A} = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{pmatrix}.$$

The matrix $k\mathbf{A}$ is a scalar multiple of the matrix $\mathbf{A}$.

General properties of matrices

For any matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of appropriate size, and any real number k:

$$\begin{array}{ll} \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} & (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \\ (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) & \mathbf{A}(k\mathbf{B}) = (k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) \\ \mathbf{AB} + \mathbf{AC} = \mathbf{A}(\mathbf{B} + \mathbf{C}) & \end{array}$$

Vectors

A **vector** is a matrix with only one column. Elements of a vector $\mathbf{v}$ are often called **components** and are specified as v_i. The size of a vector is the number of components it has.

For vectors $\mathbf{u}$ and $\mathbf{v}$, the vectors $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ are formed according to the definitions for general matrices.

Population modelling

A matrix model for the structure of a population in terms of two interdependent subpopulations J_n and A_n is given by

$$\mathbf{p}_{n+1} = \mathbf{M}\mathbf{p}_n \quad (n = 0, 1, 2, \ldots),$$

where $\mathbf{M}$ is a 2×2 matrix and $\mathbf{p}_n$ is the vector $\begin{pmatrix} J_n \\ A_n \end{pmatrix}$ which gives the subpopulation sizes at n years after a chosen starting time. The closed-form solution for this model is

$$\mathbf{p}_n = \mathbf{M}^n\mathbf{p}_0 \quad (n = 1, 2, 3, \ldots).$$

Inverting 2 × 2 matrices

The matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2×2 **identity matrix**. For any 2×2 matrix $\mathbf{A}$,

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}.$$

If two 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ have the property that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then $\mathbf{B}$ is the **inverse of A**. The inverse of a matrix $\mathbf{A}$ is usually denoted $\mathbf{A}^{-1}$. The inverse of the general 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \text{provided } ad - bc \neq 0.$$

The **determinant** of a square matrix is a number calculated from its elements. For a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is given by $\det \mathbf{A} = ad - bc$.

Determinant test for invertibility

If the determinant of a matrix $\mathbf{A}$ is not zero, then $\mathbf{A}$ has an inverse and $\mathbf{A}$ is **invertible**.

If the determinant of a matrix $\mathbf{A}$ is zero, then $\mathbf{A}$ does not have an inverse and $\mathbf{A}$ is **non-invertible**.

Solving a pair of simultaneous linear equations using matrices

Write the simultaneous linear equations in matrix form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is the coefficient matrix, $\mathbf{x}$ is the vector of variables and $\mathbf{b}$ is the vector with components equal to the right-hand sides of the equations.

If the matrix $\mathbf{A}$ is invertible, then the solution is given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

MST121 Chapter B3 Modelling with vectors

Vectors can be represented by 2×1 matrices (column form), arrows in the (x, y)-plane (geometric form), or written in terms of the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ (component form).

Arithmetic of vectors in column form

The **sum** of two vectors with the same number of components is formed by adding the corresponding components.
The **scalar multiple** of a vector $\mathbf{a}$ by a real number (scalar) k, denoted by $k\mathbf{a}$, is formed by multiplying each component of $\mathbf{a}$ by k.

Arithmetic of vectors in geometric form

Triangle Rule

To find the sum $\mathbf{a} + \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ in geometric form.

1. Choose any point P in the plane.
2. Draw an arrow to represent $\mathbf{a}$, with tail at P and tip at Q, say.
3. Draw an arrow to represent $\mathbf{b}$, with tail at Q and tip at R, say.
4. Draw the arrow with tail at P and tip at R, to complete the triangle PQR. This last arrow represents the vector $\mathbf{a} + \mathbf{b}$.

Parallelogram Rule

To find the sum $\mathbf{a} + \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ in geometric form.

1. Choose any point P in the plane.
2. Draw an arrow to represent $\mathbf{a}$, with tail at P and tip at Q, say.
3. Draw an arrow to represent $\mathbf{b}$, with tail at P and tip at S, say.
4. Complete the parallelogram $PQRS$, and draw the arrow with tail at P and tip at R. This last arrow represents the vector $\mathbf{a} + \mathbf{b}$.

Scalar multiplication

If $\mathbf{a}$ is a vector in geometric form and k is a real number, then the scalar multiple $k\mathbf{a}$ has magnitude $|k\mathbf{a}| = |k||\mathbf{a}|$. If k is non-zero, then the direction of $k\mathbf{a}$ is the same as that of $\mathbf{a}$ if $k > 0$, or opposite to that of $\mathbf{a}$ if $k < 0$.

Arithmetic of vectors in component form

The **sum** of two vectors in component form, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, is given by $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}$.

The **scalar multiple** of a vector in component form, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, by a real number k, is given by $k\mathbf{a} = ka_1\mathbf{i} + ka_2\mathbf{j}$.

Converting vectors from geometric form to component form

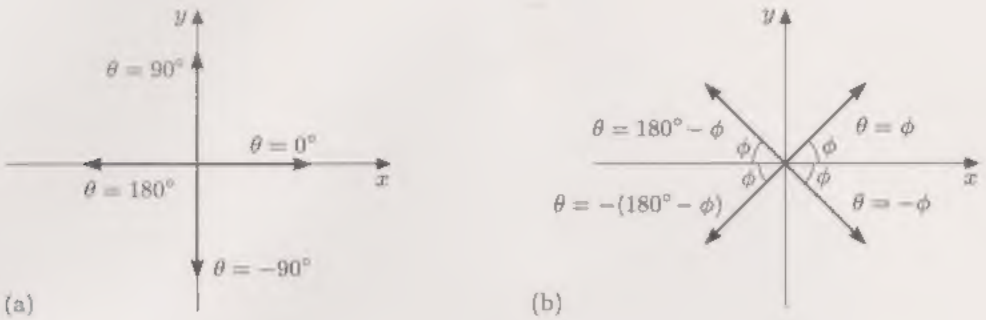
A vector $\mathbf{a}$ in geometric form, with direction θ, has component form

$$\mathbf{a} = |\mathbf{a}|\cos\theta\,\mathbf{i} + |\mathbf{a}|\sin\theta\,\mathbf{j};$$

that is, the $\mathbf{i}$-component of $\mathbf{a}$ is $a_1 = |\mathbf{a}|\cos\theta$ and the $\mathbf{j}$-component of $\mathbf{a}$ is $a_2 = |\mathbf{a}|\sin\theta$.

Converting vectors from component form to geometric form

A vector in component form, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, has magnitude $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$. If the vector is non-zero, then its direction, in terms of the angle θ measured anticlockwise from the positive x-axis, is obtained from the figures below.



(a) $\mathbf{a}$ is parallel to a coordinate axis (b) $\phi = \arctan(|a_2/a_1|)$

Sine and Cosine Rules

By convention, in a triangle, the vertex labels A, B and C are also used to denote the corresponding angle sizes, while the side lengths opposite the angles are denoted by a, b and c, respectively.

In a triangle ABC, if $a < b$ then $A < B$, and vice versa.

Sine Rule

For any triangle, the side lengths a, b, c and corresponding opposite angles A, B, C are related by the formulas

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or, equivalently,} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Cosine Rule

For any triangle, the side lengths a, b, c and corresponding opposite angles A, B, C are related by the formulas

$$a^2 = b^2 + c^2 - 2bc\cos A, \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$b^2 = c^2 + a^2 - 2ca\cos B, \qquad \cos B = \frac{c^2 + a^2 - b^2}{2ca},$$

$$c^2 = a^2 + b^2 - 2ab\cos C, \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Equilibrium Condition for forces

If an object is acted upon by n forces, $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$, and remains at rest in the absence of other forces, then the force vectors satisfy the equation

$$\sum_{i=1}^{n} \mathbf{F}_i = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \mathbf{0}.$$

Printed in the United Kingdom by
Lithmark Limited